METROLOGY AND MEASUREMENT SYSTEMS
VOL. XIV, NUMBER 2 (2007)

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ON IMPROVEMENT OF SAMPLING TECHNIQUE DESIGNED FOR PRECISE ROOT MEAN SQUARE MEASUREMENTS

A method for precise root mean square (RMS) measurement of periodic signals based on signal reconstruction is analysed. The RMS value of a signal under test is determined in three steps. In the first step, the Fourier coefficients are estimated from integrative samples, in the second step the estimators are corrected, using the method of least squares with constraints and precision measurement of rectified signal average, and in the third step the RMS value is calculated from the Fourier coefficient estimators. Properties of integrative samples are analysed and the problem of choosing the optimal integration time is discussed. It is shown that this approach, optimal reconstruction with correction, considerably increases the accuracy of the RMS measurements of low frequency signals.

Keywords: RMS measurements, signal reconstruction, integrative sampling

1. INTRODUCTION

Presently the most accurate RMS measurements are made by systems based on thermal voltage converters (TVC), and relative measurement uncertainties in standard laboratories can be as low as 1 μV/V [1]. However, instruments based on TVC's have many disadvantages: insufficient thermal inertia for low frequency signals, they are sensitive to overloading and very expensive. Investigation made by Pogliano [1, 2] shows that the uncertainty of the TVC can drop, due to thermal inertia, from 1 ppm level for 1000 Hz frequency signals to 10 ppm level for 10 Hz signals. Additionally, we can observe steady improvement of analogue to digital converter (ADC) parameters. Their uncertainty can be as high as 1 ppm [2], and we can expect that it will be still improving during forthcoming years. Then it is reasonable to develop a sampling technique for precise measurements of signal parameters. Two kinds of samples can be used: point samples and integrative samples. It has been shown that the latter one is particularly advantageous for precise measurement of low frequency signals [1, 3]. It provides very accurate measurements using the reconstruction technique with correction [3]. The correction improves significantly the accuracy of determination of the RMS value, but the final results depend also on reconstruction accuracy and this can be improved by optimization of integrative samples.

In the paper we investigate the possibility to apply high accuracy integral samples obtained by using an ADC to reconstruct the signal under test and to determine in this way its parameters, particularly the RMS value. Properties of integrative samples are discussed and their application for periodic signal reconstruction is analyzed. The very important problem of choosing the optimal integration time is discussed and three possible solutions are proposed. General formulas for estimators of the Fourier series coefficients and the RMS value of the signal are derived by using the method of least squares (LS) based on integrative samples. Additionally, a method for correction of the estimators, based on rectified signal average (RSA) measurements is presented. The effectiveness of correction is analyzed for some important signals and its value assessed. It is shown that the method makes it possible to obtain a significant improvement of RMS measurement accuracy by using commercially available apparatus.

2. PROBLEM STATEMENT

Let the signal under test, $s(t)$, be periodic with period T and band limited, then it can be expanded into a finite Fourier series

$$s(t) = a_0 + \sum_{k=1}^{K} [a_k \cos(\omega k t) + b_k \sin(\omega k t)], \tag{1}$$

where $\omega = 2\pi f$ is the basic angular frequency and $f = T$. If we sample the signal $s(t)$ at the rate f_s samples per second and collect N points, we get the data sequence $\{s_n\}$, $n = 0, 1, \ldots, N-1$. Further considerations are also valid for the general case, when the samples are not uniformly displaced and are taken at any instances t_0, t_1, ..., t_{N-1}. The aim of the reconstruction is to estimate the Fourier coefficients a_0, a_k, b_k, $k = 1, 2, ..., K$, and signal frequency, f, from samples. Having estimated the values of the Fourier coefficients it is possible to estimate signal parameters, particularly its RMS value. To avoid aliasing, the sampling frequency $f_s = 1/T_s$, should satisfy the Nyquist inequality $f_s > 2Kf$.

The number of parameters a_0, a_1, ..., a_K, b_1, b_2, ..., b_K, to be estimated is equal to $2K + 1$, therefore to estimate it, the number of samples, N, should be equal or greater than 2 $(K + 1)$, i.e. $N \geq 2\ (K + 1)$. The number K, of the highest Fourier series coefficient, is usually unknown and should be also determined from samples. The value of K can be determined from samples by investigating the relation between samples and the reconstructed signal.

When estimates of the Fourier coefficients are known they can be used for calculation of the RMS value, rms, by using the formula

$$rms = \sqrt{a_0^2 + \frac{1}{2}\sum_{k=1}^{K}\left(a_k^2 + b_k^2\right)}. \tag{2}$$

To obtain high accuracy of reconstruction instead of point samples, integrative samples are used and additionally the rectified signal average (RSA) of the signal is precisely measured. These results can be used for correcting the estimates of the Fourier coefficients. A proper formula for correction is derived using the method of least squares with constraints.

3. INTEGRATIVE SAMPLES

We shall discuss now properties of integrative samples, in particular their frequency characteristics. The n-th integrative sample $s_n = s_a[nT_s]$ of the signal $s(t)$ at the instant $t_n = nT_s$, where T_s is the sampling period, is determined by the formula

$$s_n = s_a[t_n] = \frac{1}{T_a}\int_{t_n - T_a/2}^{t_n + T_a/2} s(t)dt, \tag{3}$$

where T_a is the integration time. If $T_a << T$, then the integrative sampling is well approximated by point sampling. If $T_a \to 0$ then the integrative sample becomes a point sample for $t = t_n$. However when T_a is a sizeable fraction of T, we can expect a low-pass filtering effect that is not negligible. We analyze this effect more precisely.

Define the auxiliary signal $y(t)$ as the definite integral of $s(t)$, that is

$$y(t) = \int_{-\infty}^{t} s(\tau)d\tau + c, \tag{4}$$

where c is an arbitrary constant. Next we define the other auxiliary signal

$$x(t) = \frac{1}{T_a}[y(t + T_a/2) - y(t - T_a/2)], \tag{5}$$

then the integrative sample $s_a[nT_s]$ of $s(t)$ at the instant $t_n = nT_s$ can be presented as a point sample $x[nT_s]$ of $x(t)$ by formula

$$s_a[nT_s] = x[nT_s] = \frac{1}{T_a}[y(nT_s + T_a/2) - y(nT_s - T_a/2)]. \tag{6}$$

Let $S(\omega)$ be the Fourier transform of $s(t)$, i.e. form the pair $s(t) \leftrightarrow S(\omega)$. If $s(t)$ is shifted in time then [5]

$$s(t-a) \leftrightarrow e^{-j\omega a}S(\omega) \tag{7}$$

and if $s(t)$ is integrated in time then

$$\int_{-\infty}^{t} s(\tau)d\tau \leftrightarrow \frac{1}{j\omega}S(\omega). \tag{8}$$

Using these properties we get

$$y(t+T_a/2) \leftrightarrow \frac{1}{j\omega}S(\omega)e^{j\omega T_a/2}. \tag{9a}$$

$$y(t-T_a/2) \leftrightarrow \frac{1}{j\omega}S(\omega)e^{-j\omega T_a/2}, \tag{9b}$$

Then by applying (9) to the signal $x(t)$, (5), and by linearity of the Fourier transform, we obtain

$$x(t) \leftrightarrow \frac{1}{j\omega T_a}S(\omega)\left[e^{j\omega T_a/2} - e^{-j\omega T_a/2}\right]. \tag{10}$$

The right hand side of (10) can be simplified by using Euler's formula $\sin(\alpha) = 0.5\ j(e^{-j\alpha} - e^{j\alpha})$ and the sampling function

$$\mathrm{sinc}(x) = \begin{cases} \dfrac{\sin x}{x} & \text{dla } x \neq 0 \\ 1 & \text{dla } x = 0 \end{cases}. \tag{11}$$

Next by applying this to (10) we get the pair

$$x(t) \leftrightarrow X(\omega) = \mathrm{sinc}\left(\frac{\omega T_a}{2}\right)S(\omega). \tag{12}$$

The Fourier transform $\hat{X}(\omega)$ of the point samples $x[nT_s]$ is given by the Eq. [5].

$$\hat{X}(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X\left(\omega - \frac{2\pi n}{T_s}\right), \tag{13}$$

then the Fourier transform $\hat{S}_a(\omega)$ of integrative samples $s_a[nT_s]$ is:

$$\hat{S}_a(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X\left(\omega - \frac{2\pi n}{T_s}\right) \tag{14}$$

and by (12) we get the pair

$$\hat{S}_a(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} \operatorname{sinc}\left[\left(\omega - \frac{2\pi n}{T_s}\right) \frac{T_a}{2}\right] S\left(\omega - \frac{2\pi n}{T_s}\right). \tag{15}$$

Finally by restricting the signal to the main lobe of the function sinc we obtain the formula:

$$\hat{S}_a(\omega) = \frac{1}{T_s} \operatorname{sinc}\left(\omega \frac{T_a}{2}\right) S(\omega). \tag{16}$$

As we see, integrative sampling is equivalent to low pass filtering (with transfer function sinc) followed by point sampling. When T_a is small with respect to the signal period T, the attenuation of the low pass filter is negligible. When T_a approaches T, this attenuation is approaching infinity. This property imposes limits on integration time and this is in contradiction to properties of A/D converters – high accuracy converters need long transformation (integration) time. However it is also possible to take advantage of this filtering property of integrative samples. This leads to the problem of choosing the optimal value of integration time T_a with respect to the accuracy of estimation and this is discussed hereafter, in section 5.

4. SIGNAL RECONSTRUCTION

By substituting (1) into (3), after transformations, we get the formula

$$s_a[t_n] = a_0 + \sum_{k=1}^{K} [a_k \operatorname{sinc}(k\theta) \cos(k\omega t_n) + b_k \operatorname{sinc}(k\theta) \sin(\omega k t_n)], \tag{17}$$

for the nth integrative sample, where $\theta = \pi(T_a/T)$ is a *relative integration time* referred to the signal period $T = 1/f$.

By defining the *transformed coefficients*

$$\begin{aligned} \alpha_0 &= a_0, \\ \alpha_k(\theta) &= a_k \operatorname{sinc}(k\theta), \\ \beta_k(\theta) &= b_k \operatorname{sinc}(k\theta), \quad k = 1, ..., K. \end{aligned} \tag{18}$$

Equation (17) can be presented in the form:

$$s_a[t_n] = \alpha_0 + \sum_{k=1}^{K} [\alpha_k(\theta) \cos(k\omega t_n) + \beta_k(\theta) \sin(k\omega t_n)], \tag{19}$$

suitable for the estimation of transformed coefficients $\alpha_k(\theta)$ and $\beta_k(\theta)$and also for the estimation of the Fourier coefficients α_k and b_k from (18). Notice that if $\theta = 0$ then $\alpha_k = a_k$ and $\beta_k = b_k$.

Let $s_n = s_a[t_n] + \Delta_n$, $n = 0, 1, \ldots, N-1$, $N > 2K + 1$ are measurement values of samples, where random errors Δ_n of measurements are unbiased, $E(\Delta_n) = 0$, have the same variance $\mathrm{var}(\Delta_n) = \sigma^2$ and are not mutually correlated. Under these assumptions the LS estimators minimize the residual sum of squares [4]:

$$Q = \sum_{n=1}^{N} \left[s_n - \left(\alpha_0 + \sum_{k=1}^{K} [\alpha_k(\theta)\cos(k\omega t_n) + \beta_k(\theta)\sin(k\omega t_n)] \right) \right]^2 , \tag{20}$$

which, by introducing the matrix notation, becomes

$$Q = (\boldsymbol{s} - \boldsymbol{X}\boldsymbol{a})^{\mathrm{T}}(\boldsymbol{s} - \boldsymbol{X}\boldsymbol{a}), \tag{21}$$

where $\boldsymbol{s} = [s_0, s_1, ..., s_{N-1}]^T$ is a vector of integrative samples,
$\boldsymbol{a} = [\alpha_0, \alpha_1(\theta), ..., \alpha_K(\theta), \beta_1(\theta), ..., \beta_K(\theta)]^{\mathrm{T}}$ is a vector of the transformed coefficients,

$$\boldsymbol{X} = \begin{bmatrix} 1 & \cos\omega t_1 & \cdots & \cos K\omega t_1 & \sin\omega t_1 & \cdots & \sin K\omega t_1 \\ 1 & \cos\omega t_2 & \cdots & \cos K\omega t_2 & \sin\omega t_2 & \cdots & \sin K\omega t_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \cos\omega t_N & \cdots & \cos K\omega t_N & \sin\omega t_N & \cdots & \sin K\omega t_N \end{bmatrix} \tag{22}$$

is the design matrix and the superscript T denotes transposition.

Assume that the frequency f is known – measured before the experiment (the case with unknown frequency can also be solved by minimizing Q, but it leads to a nonlinear problem and will not be considered here). If the matrix $\mathbf{X}$ has full rank, the least squares estimators $\hat{\alpha}_k(\theta)$ and $\hat{\beta}_k(\theta)$, $k = 1, ..., K$, of the transformed coefficients are [4]

$$\hat{\boldsymbol{\alpha}} = (\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{X})^{-1}\boldsymbol{X}^{\boldsymbol{T}}\boldsymbol{s}. \tag{23}$$

Next, by (18), we get estimators $\hat{a}_0 = \hat{\alpha}_0$, $\hat{a}_k = \hat{\alpha}_k/\mathrm{sinc}(k\theta)$ and $\hat{b}_k = \hat{\beta}_k/\mathrm{sinc}(k\theta)$, $k = 1, ..., K$, of the Fourier coefficients, which by substituting in (1) provide the reconstructed signal $\hat{s}(t)$.

5. CHOICE OF INTEGRATION TIME

As shown in section 3 the integrative samples have filtering properties and the noise filtering characteristic is improving when the integration time is getting longer. In addition, very accurate analogue to digital converters need a relatively long transformation time and usually their transformation accuracy is improving with increasing transformation (integration) time. These two factors advocate for applying as long integration time as possible. However, this is in contradiction with the property of the transformation function, sinc, as can be observed in Eqs. (16) and (18), which transfer regular Fourier coefficients a_k and b_k to transformed Fourier coefficients α_k and β_k. To carry out this transformation the function sinc($k\theta$) in (18) cannot be equal to zero, and to maintain high accuracy should not be also too close to zero. This means that the integration time T_a cannot be equal (or close) to a multiple of the signal under test period T, and this requirement should be satisfied for all Fourier series harmonics i.e.

$$kT_a/T \notin N, \; k = 1, \; 2, \; ..., \; K, \tag{24}$$

where $\boldsymbol{N}$ is a natural number. The condition (24) can be accomplished using one of three methods which will be discussed shortly hereafter.

Method 1 (*main lobe method*)

The simplest way to satisfy (24) has been proposed by Pogliano [1], who assumed that all points $k\theta$, $k = 1, \ldots, K$, are within the main lobe of sinc($k\theta$), and from this we get the following upper bound for integration time T_a:

$$T_a < T/K. \tag{25}$$

However this condition is very strict and limits the application of the method to low frequency signals only – when T is large, and simultaneously of very low distortion – when the number of the highest harmonic K is small. Pogliano applied the main lobe method to very pure sinusoidal signals with frequencies under 20 Hz.

Method 2 (*alternating method*)

The *alternating method* makes it possible to increase the integration time T_a beyond the limit (25) and still satisfy the condition (24), by choosing T_a for which values of $k\theta$, $k = 1, \ldots, K$, are outside the main lobe of sinc($k\theta$) and simultaneously alternate with zeros of these functions. In practical applications the functions sinc($k\theta$) not only should not be equal to zero, but also, to preserve high accuracy, should not be too small for all values of $k = 1, \ldots, K$. We can try to determine the optimal value of θ, which is big enough (integration time T_a is long) and simultaneously minimizes the transformation error. But to accomplish this, many specifications should be known prior to experiment, as for example: (*a*) the relation between the conversion error of the analogue-to-digital converter and its conversion (integration) time T_a, (*b*) the number K of Fourier series harmonics and (*c*) the values of Fourier coefficients. This is seldom the case, then we propose the following universal approach.

Assuming that $T_a > 0.5\ T$ ($\theta = \pi T_a/T > \pi/2$) we choose such a value of the integration time T_a for which the multiple

$$M_K(\theta) = \max_{\theta > \pi/2} \left\{ \prod_{k=1}^{K} |\mathrm{sinc}(k\theta)| \right\}, \quad K = 1, 2, ... \tag{26}$$

of absolute values of sinc($k\theta$) is maximal. An analysis of the functions $M_K(\theta)$ for various K has been performed using a special program written in MATLAB. A typical result of this analysis, a plot of $M_K(\theta)$ versus θ for $K = 5$ and for interval $1/2 < T_a/T < 1$ is presented in Fig. 1. It can be observed that the multiple $M_5(\theta)$ has local maxima and zeros, and the latter values of θ should be avoided. We confined the presentation of $M_K(\theta)$ to the interval $1/2 < T_a/T < 1$ because investigation shows that outside this region, i.e. when $T_a/T > 1$, the value of $M_K(\theta)$ is significantly smaller then in the interval $1/2 < T_a/T < 1$, and this situation is not suitable and should be avoided. The function $M_5(\theta)$ has five local maxima and one global maximum for $T_a/T \cong 0.555$ and the optimal value of integration time in the interval <0, 5T, T> is $T_a \cong 0.555\ T$.

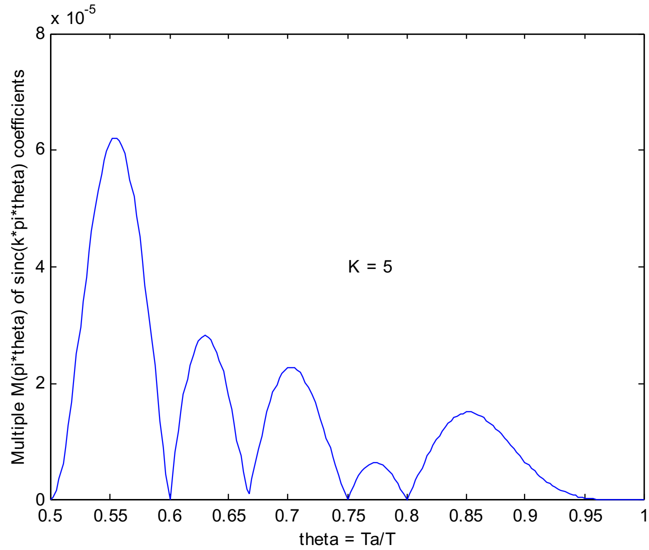

Fig. 1. Plot of $M_K(\theta)$ = M(pi*theta) of coefficients sinc($k\theta$) = sinc(k*pi*theta) for $K = 5$; relative integration time theta = T_a/T.

General analysis of plots $M_K(\theta)$, performed for different values of K revels that the functions have many local maxima, see Table 1, and the number of the maxima is greater than the number of harmonics K, when $T_a \in$<0, 5T, T>. For all K the lowest local maximum, counting from $T_a/T = 0.5$, is also the global maximum (see Fig. 1, where there is an example for $K = 5$). A global maximum indicates the optimum value for integration time T_a. All neighbourhood local maxima are separated by zeros of $M_K(\theta)$, then the value of T_a should be determined accurately enough to avoid to be too close to zero, and this is particularly important when K is getting large. Optimal values of integration time T_a, for $K = 1, 2, \ldots, 10$, are presented in Table 1 in reference to period T of the signal under test, where we can also observe that the optimal values of T_a are usually slightly greater than 0.5 T, and for $K = 1, \ldots, 10$ they are from the interval <0.535 T, 0.648 T>.

Table 1. Optimal value $(T_a/T)_{\max}$ of relative integration time as a function of the maximal number K of harmonics; $M_{\max}$ is a maximal value of $M_K(\theta)$ and N is the number of maxima of $M_K(\theta)$.

K	1	2	3	4	5	6	7	8	9	10
$(T_a/T)_{\max}$	0.500	0.648	0.570	0.586	0.555	0.560	0.543	0.545	0.535	0.538
$M_K(\theta)$	0.64	0.088	0.001	1×10^{-3}	6×10^{-5}	5×10^{-6}	2×10^{-7}	1×10^{-5}	3×10^{-10}	2×10^{-11}
] N	1	1	2	3	5	6	9	11	14	>15

If the number K of the highest harmonic is unknown, we recommend to set the value of T_a from the low part of this region i.e. $T_a \approx 0.54\ T$. Since with increasing values of K maxima, and zeros separated from them are getting denser, then values of T_a should be set with increasing accuracy, when K is becoming large, and for $K > 10$ exactly calculated and set. From this follows the conclusion that application of the alternating method for strongly distorted signals should be used only when the optimal value of T_a is exactly determined and set.

By using the alternating method we can increase the integration time five to ten times in comparison to the main lobe method.

Method 3 (*sequential method*)

Another way of increasing integration time T_a is by using a *sequential method* in which harmonics are estimated separately in groups, choosing the integration time suitable for each group. A detailed description of the sequential method is outside the scope of this consideration and we present only a general concept of the idea here.

In the first step of the sequential method we choose the integration time equal to a multiple of the signal period T, then

$$T_a = m\,T,$$

where $m \in \boldsymbol{N}$. For this value of T_a the function $\mathrm{sinc}(k\theta)$ is equal to zero, $\mathrm{sinc}(k\theta) = 0$, for all $k = 1, \ldots, K$ and, by (3), the integration sample s_0 is simply a direct measurement of the constant component a_0 and can be treated as its estimator, i.e. $\hat{a}_0 = s_0$. Then a single sample is enough to estimate the constant component a_0.

In the second step of the sequential method we choose the integration time equal to

$$T_a = (0.5 + m)\,T,$$

where $m \in \boldsymbol{N}$. For these values of T_a, the transfer functions $\mathrm{sinc}(\pi k\theta)$ are equal to zero, $\mathrm{sinc}(\pi k\theta) = 0$, for all even k, and in (17) only odd components remain. Then, taking a proper number of such integrative samples $s_1, \ldots, s_n$ we can transform them by subtracting s_0

$$s_i' = s_i - s_0, \quad i = 1, \ldots, n,$$

and next using the transformed samples and formula (23), we obtain estimators for odd Fourier coefficients a_{2l+1} and b_{2l+1}, $l = 1, 2, \ldots$.

For estimating even Fourier coefficients we chose another integration time, for example

$$T_a = (1/3 + m)T$$

and take a proper number of samples. These samples should be transformed using estimated values of the constant component and odd components of Fourier coefficients, and next used in formula (23) to estimate even harmonics, except those which are multiple of three. Continuing this process for other properly chosen values of T_a we estimate the remaining components of the Fourier coefficients.

By using the sequential method it is possible to increase the integration time almost freely, provided it is adjusted accurately to assumed values, suitable for the currently estimated group of Fourier coefficients. The sequential method is recommended for signals with a very stable frequency, which in precise measurements is usually the case.

6. CORRECTION OF RECONSTRUCTED SIGNAL

Using a precise integral analog-to-digital converter it is possible to measure, in addition to sampling, the rectified signal average (RSA)

$$rsa(s(t)) = \frac{1}{T}\int_{t=0}^{T} |s(t)|dt, \tag{27}$$

of the signal $s(t)$ with much higher accuracy than that of measured samples (for measuring the RSA value we can use a long integration time). This opens the possibility to improve the results of reconstruction by correcting the estimates of Fourier coefficients. Due to sampling errors the RSA value, $rsa(\hat{s}(t))$, of the reconstructed signal $\hat{s}(t)$ is different from $rsa(s(t))$ and its measured value $s_A = rsa(s(t)) + \Delta_A$, where the error Δ_A is much smaller than errors of the integrative samples. It is reasonable to adjust the Fourier coefficients of the reconstructed signal to obtain the *corrected signal* $\hat{s}_c(t)$ which satisfies the equation

$$rsa(\hat{s}_c(t)) = rsa(s(t)). \tag{28}$$

The correction can be performed by using the LS method with constraints [4]. Further analysis will be performed for signals with low distortion. More specifically we assume that the signal $s(t)$ satisfies the conditions:

a) $s(t) = 0$ for $t = 0$,

b) $s(t) \geq 0$ for $t \in <0, T_0>$ and $s(t) < 0$ for $t \in <T_0, T>$,

where $T_0 \in <0, T>$. For such signals the RSA value is

$$rsa(s(t)) = \frac{1}{T}\left(\int_0^{T_0} s(t)dt - \int_{T_0}^{T} s(t)dt\right). \qquad (29)$$

By substituting the Fourier series (1) in (29), after transformations, we obtain

$$rsa(s(t)) = a_0\left(\frac{\theta_0}{\pi} - 1\right) + \sum_{k=1}^{K}\left[\frac{a_k}{\pi k}\sin(k\theta_0) + \frac{b_k}{\pi k}(1 - \cos(k\theta_0))\right], \qquad (30)$$

where $\theta_0 = 2\pi(T_0/T)$. The important property of the Eq. (30) is a linear relation of $rsa(s(t))$ on the Fourier coefficients $a_0, a_1, \ldots, a_K, b_1, \ldots, b_K$. By (18) we substitute in (30) into the transformed coefficients $\alpha_k(\theta)$ and $\beta_k(\theta)$ then

$$rsa(s(t)) = \alpha_0\left(\frac{\theta_0}{\pi} - 1\right) + \sum_{k=1}^{K}\left[\frac{\alpha_k(\theta)}{\pi k}\frac{\sin(k\theta_0)}{\text{sinc}(k\theta)} + \frac{\beta_k(\theta)}{\pi k}\frac{(1 - \cos(k\theta_0))}{\text{sinc}(k\theta)}\right]. \qquad (31)$$

If we introduce the matrix notation

$$Z = \left[\frac{\theta_0}{\pi} - 1, \frac{1}{\pi}\frac{\sin(\theta_0)}{\text{sinc}(\theta)}, \ldots, \frac{1}{\pi K}\frac{\sin(K\theta_0)}{\text{sinc}(K\theta)}, \frac{1}{\pi}\frac{1 - \cos(\theta_0)}{\text{sinc}(\theta)}, \ldots, \frac{1}{\pi K}\frac{1 - \cos(K\theta_0)}{\text{sinc}(K\theta)}\right], \qquad (32)$$

then Eq. (31) can be written in a compact form

$$rsa(s(t)) = \boldsymbol{Za}. \qquad (33)$$

The corrected estimates $\hat{\mathbf{a}}_c$ of the transformed coefficients can be determined by minimizing the residual sum of squares Q under the constraint Eq. (33) [4]. This is a problem of finding a conditional minimum, whose solution, by the method of Lagrange multipliers, see [3], is

$$\hat{\boldsymbol{a}}_c = \hat{\boldsymbol{a}} + corr = \hat{\boldsymbol{a}} + \frac{rsa(s(t)) - \boldsymbol{Z}\hat{\boldsymbol{a}}}{\boldsymbol{Z}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{Z}^T}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{Z}^T, \qquad (34)$$

where $\hat{\boldsymbol{a}}$ is the vector of the transformed Fourier coefficient estimators given by (23). In practical situations we substitute in (34) the measured value s_A for $rsa(s(t))$. The estimator $\hat{\boldsymbol{a}}_c$ is equal to the sum of two elements: estimates $\hat{\boldsymbol{a}}$ and the correction term $corr$. The latter one depends on the difference $rsa(s(t)) - \boldsymbol{Z}\hat{\boldsymbol{a}}$, where $\boldsymbol{Z}\hat{\boldsymbol{a}} = rsa(\hat{s}(t))$.

By substituting the corrected estimates $\hat{\boldsymbol{a}}_c$ in (1) we obtain the corrected signal $\hat{s}_c(t)$ and by substituting $\hat{\boldsymbol{a}}_c$ in (2) we obtain the corrected value of RMS.

7. EFFECTIVENESS OF THE CORRECTION

It is important to asses the improvement in accuracy of the reconstructed signal attained by the correction. Each Fourier coefficient is corrected by (34) to a different degree then an aggregate measure of improvement is needed. Referring to the RMS we define the *effectiveness coefficient eff* as the ratio, $eff = \delta_r/\delta_c$, where δ_r and δ_c are relative errors of the RMS value determination from the reconstructed signal, $\hat{s}(t)$, and the corrected signal, $\hat{s}_c(t)$, respectively. The value of *eff* indicates how many times the error is suppressed by the correction.

It is a complex task to derive a general formula for *eff* valid for any periodic signal, then we present two important cases using point sampling (integrative sampling with $\theta = 0$).

Case 1. Correction of the signal $s(t) = b_1 \sin(\omega t)$

Assume that after the reconstruction we get the erroneous estimate $\hat{\boldsymbol{a}} = \hat{b}_1 \neq b_1$ and that the RSA value is known exactly: $s_A = rsa(s(t)) = (2/\pi)b_1$ (the measurement error of s_A is $\Delta_A = 0$). The elements of the constraint equation are: $\boldsymbol{Z} = [2/\pi]$, $\boldsymbol{a} = [b_1]$. The design matrix $\boldsymbol{X}$ has only one column, then the term $\boldsymbol{X}^T\boldsymbol{X}$ is a number. In Equation (34) $\boldsymbol{X}^T\boldsymbol{X}$ occurs twice in the numerator and in the denominator and reduces – as a result the correction term *corr* does not depend on $\boldsymbol{X}$ and is equal to $corr = b_1 - \hat{b}_1$, then

$$\hat{\boldsymbol{a}}_c = \hat{\boldsymbol{a}} + corr = \hat{b}_1 + (b_1 - \hat{b}_1) = b_1. \tag{35}$$

Thus we get the correct result independently of the estimate $\hat{b}_1$ and the effectiveness coefficient is $eff = \infty$.

Case 2. Correction of the signal $s(t) = b_1 \sin(\omega t) + b_{2p+1}\sin[(2p+1)\omega t]$

There are two coefficients $\boldsymbol{a} = [b_1, b_{2p+1}]^{\mathrm{T}}$ to estimate, p is a natural number. By setting $\alpha = \omega t$ we turn to the angular form $s(\alpha) = b_1 \sin\alpha + b_{2p+1}\sin((2p+1)\alpha)$. It is enough to sample $s(t)$ in the interval $<0, T/2>$ (or in angular measure $<0, \pi>$), i.e. in half of the signal $s(t)$ period. $T_0 = T/2$ then $\theta_0 = 2\pi\, T_0/T = \pi$. We take N samples at $0, (1/N)\pi, (2/N)\pi, \ldots, ((N-1)/N)\pi$. Then the design matrix is

$$\boldsymbol{X} = \begin{bmatrix} \sin(0\pi/N) & \sin(0(2p+1)\pi/N) \\ \sin(1\pi/N) & \sin(1(2p+1)\pi/N) \\ \cdots & \cdots \\ \sin((N-1)\pi/N) & \sin((N-1)(2p+1)\pi/N) \end{bmatrix}. \tag{36}$$

It can be proved that if the number of samples N is a multiple of 6, then

$$\boldsymbol{X}^T\boldsymbol{X} = \frac{N}{2}\boldsymbol{I}_2 \;\Rightarrow\; (\boldsymbol{X}^T\boldsymbol{X})^{-1} = \frac{2}{N}\boldsymbol{I}_2, \tag{37}$$

where $\boldsymbol{I}_2$ is a 2×2 unit matrix (this equation can easily checked for $N = 6$). By (32) we obtain

$$\boldsymbol{Z} = \left[\frac{1}{\pi}(1 - \cos\pi)\ \frac{1}{(2p+1)\pi}(1 - \cos(\pi(2p+1)))\right] = \frac{2}{\pi}\left[1\ \ \frac{1}{2p+1}\right] \tag{38}$$

and by (33)

$$rsa(s(t)) = \frac{2}{\pi}\left(b_1 + \frac{1}{2p+1}b_{2p+1}\right). \tag{39}$$

After substituting these results in (34), after transformation, we obtain the following formulas for corrected harmonics

$$\hat{b}_1^c = b_1 - \frac{1}{1+(2p+1)^2}(b_1 - \hat{b}_1) + \frac{2p+1}{1+(2p+1)^2}(b_{2p+1} - \hat{b}_{2p+1}), \tag{40}$$

$$\hat{b}_{2p+1}^c = b_{2p+1} - \frac{(2p+1)^2}{1+(2p+1)^2}(b_{2p+1} - \hat{b}_{2p+1}) + \frac{2p+1}{1+(2p+1)^2}(b_1 - \hat{b}_1). \tag{41}$$

For not strongly distorted signals, $b_{2p+1} << b_1$, the third term of the right hand side of (40) is negligible and the corrected estimate of b_1 is about $(2p+1)^2$ times closer to the true value that the reconstructed value.

From (40) and (41) we get the relations

$$\delta_1^c = \frac{1}{1+(2p+1)^2}\delta_1 - \frac{2p+1}{1+(2p+1)^2}\frac{b_{2p+1}}{b_1}\delta_{2p+1}, \tag{42}$$

$$\delta_{2p+1}^c = \frac{(2p+1)^2}{1+(2p+1)^2}\delta_{2p+1} - \frac{2p+1}{1+(2p+1)^2}\frac{b_1}{b_{2p+1}}\delta_1, \tag{43}$$

between relative errors $\delta_1 = (\hat{b}_1 - b_1)/b_1$ and $\delta_{2p+1} = (\hat{b}_{2p+1} - b_{2p+1})/b_{2p+1}$ of the reconstructed coefficient estimates $\hat{b}_1$ and $\hat{b}_{2p+1}$, and relative errors $\delta_1^c = (\hat{b}_1^c - b_1)/b_1$ and $\delta_{2p+1}^c = (\hat{b}_{2p+1}^c - b_{2p+1})/b_{2p+1}$ of the corrected coefficients $\hat{b}_1^c$ and $\hat{b}_{2p+1}^c$. For signals with low distortion $eff \cong 1 + (2p+1)^2$ and for the least profitable case, $p = 1$, $eff \cong 10$ (for $p = 2$, $eff \cong 25$). Thus the accuracy of b_1 estimation by the correction is improved at least ten times, and approximately the same is the improvement of the *rms* determination by (2).

8. CONCLUSION

The presented method is designed for signal reconstruction and for very accurate RMS measurements of periodic signals. It is based on integrative sampling, and precise measurement of the RSA value used for correction of Fourier coefficient estimators. The best results are obtained when the integration time is optimised by using one of the proposed methods of choosing its value. The analysis shows that the accuracy of RMS measurements made by commercially available apparatus and by using the proposed estimators correction can be improved in this way as much as ten times. An important advantage of the method is that its accuracy will increase together with improvement of technology of analog to digital converters. This approach can be applied for precise measurement of other important quantities as power and energy.

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